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Projectively full radical ideals in integral extension rings

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Abstract

Let R be a Noetherian commutative ring with unit $1 \neq 0$, and let I be a regular proper ideal of R. The main question considered in this paper is whether there exists a finite integral extension ring A of R for which the nilradical of IA is a projectively full ideal that is projectively equivalent to IA. A related and stronger question that we also consider is whether there exists a finite integral extension ring A of R for which the nilradical J of IA is projectively equivalent to IA and for which all the Rees integers of J are one. The following two results are special cases of the main theorems in the present paper: (1) If R is a Noetherian integral domain, then there exists a finite integral extension ring A of R such that the nilradical of IA is projectively equivalent to IA. (2) If also R contains a field of characteristic zero, then there exists a finite free integral extension ring A of R for which the nilradical of IA is a projectively full ideal that is projectively equivalent to IA.

Keywords: Integrally closed ideal; Projectively full ideal; Projectively equivalent ideals; Rees valuations of an ideal; Rees integers; Finite free integral extension ring

1. Introduction

All rings in this paper are commutative with a unit $1 \neq 0$. Let I be a regular proper ideal of the Noetherian ring R (that is, I contains a regular element of R and $I \neq R$). An ideal J in R is *projectively equivalent* to I in case $(J^j)_a = (I^i)_a$ for some positive integers i and j

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(where K_a denotes the integral closure in R of an ideal K of R). The concept of projective equivalence of ideals and the study of ideals projectively equivalent to I was introduced by Samuel in [20] and further developed by Nagata in [13]. Making use of interesting work of Rees in [19], McAdam, Ratliff, and Sally in [11, Corollary 2.4] proved that the set $\mathbf{P}(I)$ of integrally closed ideals projectively equivalent to I is discrete and linearly ordered by inclusion. If K is an ideal in R such that every element of $\mathbf{P}(I)$ is the integral closure of a power of K, then the ideal K and the set $\mathbf{P}(I)$ are said to be *projectively full*. This definition first appeared in [1]. A number of results about, and examples of, projectively full ideals are given in [1–4]. Several characterizations of such ideals are given in [1, (4.11) and (4.12)]. In [2, Section 3] relations between projectively full ideals in R and in factor rings of R, localizations of R, and extension rings of R are proved. A main goal in these papers, and also in the present paper, is to determine conditions in order that the following question have an affirmative answer:

Question 1.1. Let I be a nonzero proper ideal in a Noetherian domain R. Does there exist a finite integral extension domain A of R such that $\mathbf{P}(IA)$ contains an ideal J whose Rees integers are all equal to one, that is, for each Rees valuation ring (V, N) of IA, $IV = N$. *(*If this holds, it then follows that $J = \mathrm{Rad}(IA)$ is a projectively full radical ideal that is projectively equivalent to IA.*)*

Some progress was made on Question 1.1 in [3]. Specifically, the main result in [3] establishes the following:

Theorem 1.2. *Let $I = (b_1, \dots, b_g)R$ and let $(V_1, N_1), \dots, (V_n, N_n)$ be the Rees valuation rings of I. Assume that $b_i V_j = IV_j$ $(= N_j^{e_j}$, say$)$ for $i = 1, \dots, g$ and $j = 1, \dots, n$, and that the greatest common divisor c of $e_1, \dots, e_n$ is a unit in R. Then $A = R[x_1, \dots, x_g] = R[X_1, \dots, X_g]/(X_1^c - b_1, \dots, X_g^c - b_g)$ is a finite free integral extension ring of R and the ideal $J = (x_1, \dots, x_g)A$ is projectively full and projectively equivalent to IA. Thus $\mathbf{P}(IA) = \mathbf{P}(J)$ is projectively full. Also, if R is an integral domain, if $z_1, \dots, z_m$ are the minimal prime ideals in A, and if $B_h = A/z_h$, then $\mathbf{P}(IB_h)$ is projectively full for $h = 1, \dots, m$.*

The two main theorems in the present paper extend Theorem 1.2. With notation as in Theorem 1.2, the first of these theorems, Theorem 2.6, shows that if $b_1, \dots, b_g$ are arbitrary regular elements in I that generate I, if m is an integer greater than or equal to $\max(\{e_i \mid i = 1, \dots, n\})$, and if $A_m = R[x_1, \dots, x_g] = R[X_1, \dots, X_g]/(X_1^m - b_1, \dots, X_g^m - b_g)$ and $J_m = (x_1, \dots, x_g)A_m$, then J_m is projectively equivalent to IA_m, $(J_m)_a = \mathrm{Rad}(J_m)$, and $A_m/(J_m)_a \cong R/\mathrm{Rad}(I)$. Further, if R is an integral domain and if z is a minimal prime ideal in A_m, then $((J_m + z)/z)_a$ is a radical ideal that is projectively equivalent to $(IA_m + z)/z$.

The ideal J_m of Theorem 2.6 may fail to be projectively full, so the second of the main results in this paper, Theorem 3.7, shows that if the generating set $b_1, \dots, b_g$ of I in Theorem 1.2 is such that $b_i V_j = IV_j$ $(= N_j^{e_j}$, say) for $i = 1, \dots, g$ and $j = 1, \dots, n$, and if the least common multiple e of $e_1, \dots, e_n$ is a unit in R, then for each positive multiple m of e that is a unit in R the ideal $(J_m)_a$ is projectively full and $(J_m)_a$ is a radical ideal that is projectively equivalent to IA_m. Also $x_i U$ is the maximal ideal of U for each Rees valuation ring U of J and for $i = 1, \dots, g$, so the Rees integers of J are all equal to one. Moreover, if R is an integral domain and if z is a minimal prime ideal in A_m, then $((J_m + z)/z)_a$ is a projectively full radical ideal that is projectively equivalent to $(IA_m + z)/z$.

A key result used in the proof of Theorem 3.7 is Proposition 3.5. This asserts that if b is an arbitrary nonzero element in the Jacobson radical of a semi-local PID $(D, M_1, \ldots, M_n)$, say $bD_{M_i} = M_i^{e_i} D_{M_i}$ for $i = 1, \ldots, n$, and if m is a positive common multiple of $e_1, \ldots, e_n$ that is a unit in D, then the integral closure E' of $E = D[b^{\frac{1}{m}}]$ in its quotient field is a semi-local PID whose Jacobson radical is $b^{\frac{1}{m}} E'$.

Corollaries 3.15 and 3.18 extend Theorem 3.7 to certain finite sets of ideals of R. Corollary 3.21 shows that Theorem 3.7 holds for each regular proper ideal I in an arbitrary Noetherian ring R that contains the rational number field. On the other hand, Example 3.22 and Remark 3.23 show that if the least common multiple of the integers $e_1, \ldots, e_n$ is not a unit in R, then the method used in the proof of Theorem 3.7 to obtain A_m and J_m does not insure that J_m is projectively full.

Our notation is mainly as in Nagata [14]. Thus a *basis* of an ideal is a generating set of the ideal, and the term *altitude* refers to what is often called dimension or Krull dimension. If R is a semi-local ring, an ideal of R is said to be *open* if it contains a power of the Jacobson radical of R.

We are indebted to the referee for several helpful suggestions. In particular, the proof of Lemma 3.1 is due to the referee; it is shorter than our original proof.

2. Projective equivalence and radical ideals

We recall several definitions that are used throughout this paper.

Definition 2.1. Let I be a regular proper ideal in a Noetherian ring R. Then:
(2.1.1) I_a denotes the *integral closure* of I in R, so I_a is the ideal $\{x \in R \mid x$ is a root of an equation of the form $X^n + i_1 X^{n-1} + \cdots + i_n = 0$, where $i_j \in I^j$ for $j = 1, \ldots, n\}$. I is *integrally closed* in case $I = I_a$.
(2.1.2) For each $x \in R$, let $v_I(x) = \max\{k \mid k$ is a nonnegative integer and $x \in I^k\}$ (as usual, $I^0 = R$). Let $v_I(x) = \infty$ in case $x \in I^k$ for all positive integers k.
(2.1.3) For each $x \in R$, let $\bar{v}_I(x) = \lim_{k\to\infty}(\frac{v_I(x^k)}{k})$ (see (2.1.2) and Remark 2.2).
(2.1.4) If k and m are positive integers, then $I_{(\frac{k}{m})}$ denotes the integrally closed ideal $\{x \in R \mid \bar{v}_I(x) \geqslant \frac{k}{m}\}$ (see (2.1.3) and [11, (2.1)(g)]).
(2.1.5) $\mathbf{P}(I)$ denotes the set of integrally closed ideals in R that are projectively equivalent to I. There exist a nonnegative integer n^* and a *unique* integer $d(I)$ such that

$$\mathbf{P}(I) \setminus \{I_{(n^* + \frac{k}{d(I)})} \mid k \text{ is a nonnegative integer}\} \subseteq \{I_{(\frac{1}{d(I)})}, \ldots, I_{(\frac{n^* d(I) - 1}{d(I)})}\}$$

(see (2.1.4), [11, (2.8) and (2.9)] and [1, (4.2)(d)]).
(2.1.6) $\mathbf{R}(R, I)$ denotes the *Rees ring of R with respect to I*, so $\mathbf{R}(R, I)$ is the graded subring $R[u, tI]$ of $R[u, t]$, where t is an indeterminate and $u = \frac{1}{t}$.
(2.1.7) Let $z_1, \ldots, z_r$ be the minimal prime ideals z in R such that $z + I \neq R$, for $i = 1, \ldots, r$ let $R_i = R/z_i$, let F_i be the quotient field of R_i, let $\mathbf{R}_i'$ be the integral closure in $F_i(u)$ of $\mathbf{R}_i = \mathbf{R}(R_i, (I + z_i)/z_i)$ (see (2.1.6)), let $p_{i,1}, \ldots, p_{i,h_i}$ be the (height one) prime divisors of $u\mathbf{R}_i'$, let $w_{i,j}$ be the valuation of the discrete valuation ring $W_{i,j} = \mathbf{R}'_{i\,p_{i,j}}$, let $e_{i,j} = w_{i,j}(u)$, let $V_{i,j} = W_{i,j} \cap F_i$, and define $v_{i,j}$ on R by $v_{i,j}(x) = w_{i,j}(x + z_i)$. Then the *Rees valuations* of I are the valuations $v_{1,1}, \ldots, v_{r,h_r}$, and the *Rees valuation rings* of I are the rings $V_{1,1}, \ldots, V_{r,h_r}$. We use

Rees I to denote the set $\{V_{i,j} \mid i = 1, \ldots, r \text{ and } j = 1, \ldots, h_r\}$ of all the Rees valuation rings of I.

(2.1.8) The *Rees integers* of I are the integers $e_{1,1}, \ldots, e_{r,h_r}$ defined by $IV_{i,j} = N_{i,j}^{e_{i,j}}$, where $(V_{1,1}, N_{1,1}), \ldots, (V_{r,h_r}, N_{r,h_r})$ are all the Rees valuation rings of I (see (2.1.7)). For fixed $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, h_r\}$, the integer $e_{i,j}$ is called the *Rees integer of I with respect to* $V_{i,j}$ or $v_{i,j}$.

(2.1.9) A *filtration* $\mathcal{F} = \{I_k\}_{k \geqslant 0}$ on R is a descending sequence of ideals I_k of R such that $I_0 = R$ and $I_h I_j \subseteq I_{h+j}$ for all nonnegative integers h and j.

Remark 2.2. Concerning these definitions, Rees shows in [19] that: (a) $\bar{v}_I(x)$ in (2.1.3) is well defined; (b) for each nonnegative integer k and for each $x \in R$, $\bar{v}_I(x) \geqslant k$ if and only if $x \in (I^k)_a$ (as usual, $(I^0)_a = R$); (c) the Rees valuations $v_{1,1}, \ldots, v_{r,h_r}$ defined on R in (2.1.7) have values in $\mathbb{N} \cup \{\infty\}$, where $\mathbb{N}$ denotes the nonnegative integers, and $v_{i,j}(x) = \infty$ if and only if $x \in z_i$; and, (d) for each $x \in R$, $\bar{v}_I(x) = \min\{\frac{v_{i,j}(x)}{e_{i,j}} \mid i = 1, \ldots, r \text{ and } j = 1, \ldots, h_r\}$, where $v_{1,1}, \ldots, v_{r,h_r}$ are the Rees valuations of I (see (2.1.7)) and $e_{1,1}, \ldots, e_{r,h_r}$ are the Rees integers of I (see (2.1.8)). (In what follows, we assume the $V_{i,j}$ and $e_{i,j}$ have been resubscripted so that Rees $I = \{(V_1, N_1), \ldots, (V_n, N_n)\}$ and the corresponding Rees integers are $e_1, \ldots, e_n$.)

The following notation will be used throughout this paper.

Notation 2.3. The symbol I denotes a regular proper ideal in a Noetherian ring R, $b_1, \ldots, b_g$ are regular elements in R that generate I, $X_1, \ldots, X_g$ are independent indeterminates, m is a positive integer, $A_m = R[x_1, \ldots, x_g]$, where for $i = 1, \ldots, g$, x_i is the residue class of X_i modulo $K_m = (X_1^m - b_1, \ldots, X_g^m - b_g)R[X_1, \ldots, X_g]$, and $J_m = (x_1, \ldots, x_g)A_m$. We call x_i a "formal" mth root of b_i. If z is a minimal prime ideal in A_m, then $b_i^{\frac{1}{m}}$ denotes the residue class modulo z of x_i; it is an mth root of $b_i + (z \cap R)$ in an algebraic closure of the quotient field of $R/(z \cap R)$, so, although mth roots are not unique, the notation is intended to suggest that we choose *this* mth root of b_i.

Remark 2.4. It is clear that A_m is a finite free integral extension ring of R of rank m^g. Also, J_m is a regular proper ideal of A_m, since IA_m is a regular proper ideal of A_m and $IA_m = (b_1, \ldots, b_g)A_m = (x_1^m, \ldots, x_g^m)A_m$, so $J_m = (x_1, \ldots, x_g)A_m \subseteq \mathrm{Rad}(IA_m)$.

It is well known [21, (1.6.1)] that if B is an integral extension ring of R such that IB is a regular ideal, then $(IB)_a \cap R = I_a$. Proposition 2.5.2 extends this to the ideals $I_{(\frac{k}{m})}$ of (2.1.4).

Proposition 2.5.

(2.5.1) *For each $x \in R$ and for all positive integers k and m it holds that $x \in I_{(\frac{k}{m})}$ if and only if $x^m \in (I^k)_a$.*[1] *In particular, $I_{(k)} = (I^k)_a$.*

(2.5.2) *Let B be an integral extension ring of R such that IB is a regular ideal. Then for all positive integers k and m it holds that $(IB)_{(\frac{k}{m})} \cap R = I_{(\frac{k}{m})}$. Also, if H is an ideal in B such that $(H^m)_a = (IB)_a$, then $(H^k)_a \cap R = I_{(\frac{k}{m})}$ for all positive integers k.*

[1] In (2.1.4) we use the definition of $I_{(\frac{k}{m})}$ given in [11, p. 391]. By (2.5.1), this definition is equivalent to the definition of $I_{(\frac{k}{m})}$ given in [21, 10.5].

(2.5.3) *For each positive integer m let $\mathbf{R}_m$ be the integral closure of $R[u, tI, u^{\frac{1}{m}}]$ in $R[u^{\frac{1}{m}}, t^{\frac{1}{m}}]$ and for each positive integer k let $I_{[\frac{k}{m}]} = u^{\frac{k}{m}}\mathbf{R}_m \cap R$. Then $\mathbf{R}_m$ is an integral extension ring of $R[u, tI]$, $I_{[\frac{k}{m}]} = I_{(\frac{k}{m})}$, $\mathbf{R}_m = R[u^{\frac{1}{m}}, t^{\frac{1}{m}} I_{(\frac{1}{m})}, t^{\frac{2}{m}} I_{(\frac{2}{m})}, \ldots]$, and $\mathcal{F}_m = \{I_{(\frac{k}{m})}\}_{k \geqslant 0}$ (with $I_{(\frac{0}{m})} = R$) is a filtration on R (see* (2.1.9)*).*
(2.5.4) *If m in* (2.5.3) *is a multiple of the integer $d(I)$ defined in* (2.1.5)*, then $\mathbf{R}_m$ is an integral extension ring of $\mathbf{R}_{d(I)}$, $\mathbf{P}(I) \subseteq \mathcal{F}_{d(I)} \subseteq \mathcal{F}_m$, and $\mathcal{F}_{d(I)} \setminus \mathbf{P}(I)$ is a finite set.*

Proof. For (2.5.1), $x \in I_{(\frac{k}{m})}$ if and only if $\bar{v}_I(x) \geqslant \frac{k}{m}$ (by (2.1.4)) if and only if $m\bar{v}_I(x) \geqslant k$ if and only if $\bar{v}_I(x^m) \geqslant k$ if and only if $x^m \in (I^k)_a$ (by Remark 2.2(b)).

For (2.5.2), let B be an integral extension ring of R such that IB is a regular ideal, let m and k be positive integers, and let H be an ideal in B such that $(H^m)_a = (IB)_a$. Then it follows as in Remark 2.4 that H is a regular proper ideal of B. Also, $(I^k)_a B \subseteq (I^k B)_a$ and $(I^k B)_a \cap R = (I^k)_a$ for all positive integers k, since B is an integral extension ring of R. With this in mind, fix $x \in R$. Then $x \in I_{(\frac{k}{m})}$ if and only if $x^m \in (I^k)_a$, by (2.5.1), if and only if $x^m \in ((IB)^k)_a \cap R = (H^{mk})_a \cap R$ (since $(IB)_a = (H^m)_a$ implies that $((IB)^k)_a = (I^k B)_a = (H^{mk})_a$) if and only if $x \in (H^k)_a \cap R$ (since, in general, $y^n \in (H^n)_a$ if and only if $y \in H_a$).

A similar proof shows that $(IB)_{(\frac{k}{m})} \cap R = I_{(\frac{k}{m})}$.

For (2.5.3), it is clear that $R[u, tI, u^{\frac{1}{m}}]$ is an integral extension ring of $R[u, tI]$ and that $\mathbf{R}_m$ is an integral extension ring of $R[u, tI, u^{\frac{1}{m}}]$, so $\mathbf{R}_m$ is an integral extension ring of $R[u, tI]$. Also, it is shown in [18, Section 4] that $\mathcal{F}_m = \{I_{(\frac{k}{m})}\}_{k \geqslant 0}$ is a filtration on R, and it is shown in [1, (4.4)] that $u^{\frac{k}{m}}\mathbf{R}_m \cap R = I_{(\frac{k}{m})}$ for all positive integers k. Therefore it follows that $\mathbf{R}_m = R[u^{\frac{1}{m}}, t^{\frac{1}{m}} I_{(\frac{1}{m})}, t^{\frac{2}{m}} I_{(\frac{2}{m})}, \ldots]$. ([21, 10.5] contains some related material.)

For (2.5.4), it is clear that $R[u, tI] \subseteq \mathbf{R}_{d(I)}$, and if m is a multiple of $d(I)$, then $\mathbf{R}_{d(I)} \subseteq \mathbf{R}_m$. Also, $\mathbf{R}_m$ is integral over $R[u, tI]$, by (2.5.3), so it follows that $\mathbf{R}_m$ is an integral extension ring of $\mathbf{R}_{d(I)}$. Finally, it is shown in [1, (4.5.1)] that if $m = d(I)$, then $\mathbf{P}(I) \subseteq \mathcal{F}_{d(I)}$ and that $\mathcal{F}_{d(I)} \setminus \mathbf{P}(I)$ is a finite set, and $\mathcal{F}_{d(I)} \subseteq \mathcal{F}_m$, since m is a multiple of $d(I)$. $\square$

Theorem 2.6 was suggested by the following theorem of Itoh [7, p. 392]: Let I be a regular proper ideal in a Noetherian ring R, let e be the least common multiple of the Rees integers of I, let $\mathbf{R} = \mathbf{R}(R, I)$ (see (2.1.6)), and let $\mathbf{R}_e$ be the integral closure of $\mathbf{R}[u^{\frac{1}{e}}]$ in $R[u^{\frac{1}{e}}, t^{\frac{1}{e}}]$. Then $u^{\frac{1}{e}}\mathbf{R}_e$ is a radical ideal by [7, p. 392]. See also [6,8], and [21, (10.5.6)(4)].

The ideal $u^{\frac{1}{e}}\mathbf{R}_e$ of Itoh's Theorem has several nice properties, but $\mathbf{R}_e$ is not an integral extension ring of R (since $u \in \mathbf{R}_e$ is transcendental over R) and $u^{\frac{1}{e}}\mathbf{R}_e$ is not projectively equivalent to $I\mathbf{R}_e$ (since $u^{\frac{1}{e}}\mathbf{R}_e$ is projectively full and $(I\mathbf{R}_e)_a \neq u^{\frac{k}{e}}\mathbf{R}_e = (u^{\frac{k}{e}}\mathbf{R}_e)_a$ for all positive integers k). We wondered if it is possible to construct an ideal J^* in a finite integral extension ring A of R that has properties similar to those of $u^{\frac{1}{e}}\mathbf{R}_e$ and that is projectively equivalent to IA. Theorem 2.6 shows that this is indeed the case, and Theorem 3.7 shows that if $b_i V_j = IV_j$ for $i = 1, \ldots, g$ and for all Rees valuation rings V_j of I, and if e is a unit in R, then J_m is a projectively full radical ideal that is projectively equivalent to IA and the Rees integers of J_m are all equal to one, so if U is a Rees valuation ring of J_m, then $J_m U$ is the maximal ideal of U.

Theorem 2.6. *Let m be an integer such that $m \geqslant \max(\{e_i \mid i = 1, \ldots, n\})$. Then, with the notation of* (2.3):

(2.6.1) $I_{(\frac{1}{m})} = \operatorname{Rad}(I)$.
(2.6.2) $(J_m^k)_a \cap R = I_{(\frac{k}{m})}$ *for all positive integers* k.
(2.6.3) J_m *is projectively equivalent to* IA_m, $(J_m)_a = \operatorname{Rad}(J_m)$, *and* $A_m/(J_m)_a \cong R/\operatorname{Rad}(I)$.
(2.6.4) *If* R *is an integral domain and if* z *is a minimal prime ideal in* A_m, *then* $((J_m + z)/z)_a$ *is a radical ideal that is projectively equivalent to* $(IA_m + z)/z$ *and* $(((J_m + z)/z)^k)_a \cap R = I_{(\frac{k}{m})}$ *for all positive integers* k.

Proof. For (2.6.1), let $x \in R$. If $x \in I_{(\frac{1}{m})}$, then $x^m \in I_a$, by Proposition 2.5.1, and $I_a \subseteq \operatorname{Rad}(I)$, so $x^m \in \operatorname{Rad}(I)$. Therefore $x \in \operatorname{Rad}(I)$, so $I_{(\frac{1}{m})} \subseteq \operatorname{Rad}(I)$.

For the opposite inclusion, let $x \in \operatorname{Rad}(I)$. Then x is in the center in R of every Rees valuation v_i of I, so $v_i(x) \geqslant 1$ for $i = 1, \ldots, n$. Now $\bar{v}_I(x) = \min\{\frac{v_i(x)}{e_i} \mid i = 1, \ldots, n\}$, by Remark 2.2(d), so $\bar{v}_I(x) \geqslant \min\{\frac{1}{e_i} \mid i = 1, \ldots, n\}$. Therefore, since $m \geqslant \max(\{e_i \mid i = 1, \ldots, n\})$, it follows that $\bar{v}_I(x) \geqslant \frac{1}{m}$, so it follows from (2.1.4) that $x \in I_{(\frac{1}{m})}$, hence $\operatorname{Rad}(I) \subseteq I_{(\frac{1}{m})}$.

For (2.6.2), $IA_m = (b_1, \ldots, b_g)A_m = (x_1^m, \ldots, x_g^m)A_m \subseteq J_m^m \subseteq ((x_1^m, \ldots, x_g^m)A_m)_a = (IA_m)_a$, so IA_m is a reduction of J_m^m. Therefore $(J_m^m)_a = (IA_m)_a$, so (2.6.2) follows from Proposition 2.5.2.

For (2.6.3), it was just shown that $(J_m^m)_a = (IA_m)_a$, so J_m is projectively equivalent to IA_m.

Since $A_m = R[X_1, \ldots, X_g]/K_m$, with $K_m = (X_1^m - b_1, \ldots, X_g^m - b_g)R[X_1, \ldots, X_g]$, and since $J_m = ((X_1, \ldots, X_g)R[X_1, \ldots, X_g] + K_m)/K_m = ((I, X_1, \ldots, X_g)R[X_1, \ldots, X_g])/K_m$, it follows that $J_m \cap R = I$ and that $A_m/J_m \cong R/I$. Therefore, since $(J_m)_a \cap R = I_{(\frac{1}{m})}$, by (2.6.2), since $I_{(\frac{1}{m})} = \operatorname{Rad}(I)$, by (2.6.1), and since $I \subseteq I_{(\frac{1}{m})}$, it follows that $R/\operatorname{Rad}(I) \cong A_m/(J_m)_a$. Therefore, since $R/\operatorname{Rad}(I)$ is a reduced ring, it follows that $(J_m)_a$ is a radical ideal, hence $(J_m)_a = \operatorname{Rad}(J_m)$.

For (2.6.4), it is readily checked that projective equivalence of ideals is preserved when passing to factor rings. Therefore, since A_m/z is a finite integral extension domain of $R/(z \cap R)$, the proof of (2.6.4) is similar to the proof of (2.6.2) and (2.6.3). □

We close this section with three remarks concerning Theorem 2.6.

Remark 2.7.
(2.7.1) With the notation of Theorem 2.6 and (2.3), let $\{y_1, \ldots, y_k\}$ be a nonempty subset of $\{x_1, \ldots, x_g\}$ and let $\{c_1, \ldots, c_k\}$ be the corresponding subset of $\{b_1, \ldots, b_g\}$. Then $((c_1, \ldots, c_k)A_m)_a = (((y_1, \ldots, y_k)A)^m)_a$, so $(c_1, \ldots, c_k)A_m$ is projectively equivalent to $(y_1, \ldots, y_k)A$.
(2.7.2) In Theorem 2.6, assume that R is a semi-local ring and that I is an open ideal. Then for all integers $m \geqslant \max(\{e_i \mid i = 1, \ldots, n\})$ it holds that A_m is a semi-local ring and $(IA_m)_{(\frac{1}{m})}$ is the Jacobson radical of A_m and is projectively equivalent to IA_m. Also, $I_{(\frac{1}{m})}$ is the Jacobson radical of R, but in general $I_{(\frac{1}{m})}$ is not projectively equivalent to I.
(2.7.3) In Theorem 2.6, assume that R is a Noetherian domain, let $m \geqslant \max(\{e_i \mid i = 1, \ldots, n\})$, let z be a minimal prime ideal in A_m, and let $B = A_m/z$, so $B \cong R[b_1^{\frac{1}{m}}, \ldots, b_g^{\frac{1}{m}}]$ (see (2.3)). Then

$$\left(\left(b_1^{\frac{k}{m}}, \ldots, b_g^{\frac{k}{m}}\right)B\right)_a \cap R = \left((b_1, \ldots, b_g)R\right)_{(\frac{k}{m})} \quad \text{for all positive integers } k.$$

Proof. For (2.7.1), it follows as in the proof of Theorem 2.6.2 that $(c_1,\dots,c_k)A_m \subseteq ((y_1,\dots,y_k)A_m)^m \subseteq ((c_1,\dots,c_k)A_m)_a$. Therefore $((c_1,\dots,c_k)A_m)_a = (((y_1,\dots,y_k)A_m)^m)_a$, hence $(c_1,\dots,c_k)A_m$ is projectively equivalent to $(y_1,\dots,y_k)A_m$.

For (2.7.2), $(J_m^m)_a = (IA_m)_a$, by the proof of (2.6.2), and $(J_m^m)_{(\frac{1}{m})} = (J_m)_a$, by Proposition 2.5.1 applied to J_m^m in place of I. Also, $(J_m)_a$ is the Jacobson radical of A_m, by Theorem 2.6.3, so it follows that $(IA_m)_{(\frac{1}{m})} = (J_m)_a$ is the Jacobson radical of A_m and is projectively equivalent to IA_m. Also, $I_{(\frac{1}{m})}$ is the Jacobson radical of R, by Theorem 2.6.1, but, in general, the Jacobson radical of a semi-local ring R is not projectively equivalent to every open ideal of R. (For example, if $(R, M = (x, y)R)$ is a regular local ring of altitude two and $e > 1$ is an integer, then it follows from [4, Example 3.1] that $(x, y^e)R$ is a projectively full open ideal that is not projectively equivalent to the projectively full ideal $(x, y)R$.)

For (2.7.3), $(J_m + z)/z = (b_1^{\frac{1}{m}},\dots,b_g^{\frac{1}{m}})B$. Also, $((b_1^{\frac{1}{m}},\dots,b_g^{\frac{1}{m}})^m B)_a = (IB)_a$, so $((b_1^{\frac{1}{m}},\dots,b_g^{\frac{1}{m}})^k B)_a \cap R = I_{(\frac{k}{m})}$, by Proposition 2.5.2. Finally, $((b_1^{\frac{1}{m}},\dots,b_g^{\frac{1}{m}})^k B)_a = ((b_1^{\frac{k}{m}},\dots,b_g^{\frac{k}{m}})B)_a$, and $I_{(\frac{k}{m})} = ((b_1,\dots,b_g)R)_{(\frac{k}{m})}$. $\square$

3. Projectively full radical ideals and integral extension rings

In this section we prove in certain cases that the radical ideal $(J_m)_a$ of Theorem 2.6 is projectively full and its Rees integers are all equal to one. For this purpose, we consider in Lemma 3.1 the behavior of the maximal ideals of a semi-local PID in an integral extension obtained by adjoining certain roots of units.

Lemma 3.1. *Let $(D, M_1,\dots,M_n)$ be a semi-local PID, let $u_1,\dots,u_g$ be units in D, let $e_1,\dots,e_g$ be positive integers that are units in D, for $i = 1,\dots,g$ let $u_i^{\frac{1}{e_i}}$ be an e_ith root of u_i in an algebraic closure of the quotient field of D, and let $E = D[u_1^{\frac{1}{e_1}},\dots,u_g^{\frac{1}{e_g}}]$. Then E is a finite integral extension domain of D, E is a semi-local PID, and M_iE is a radical ideal for $i = 1,\dots,n$.*

Proof. Using induction on g, it suffices to prove this lemma in the case $g = 1$, since the elements $u_2,\dots,u_g$, and the integers $e_2,\dots,e_g$, are units in $D[u_1^{\frac{1}{e_1}}]$.

For this case, $D[X]$ is a UFD and $E \cong D[X]/(f(X)D[X])$, where $f(X)$ divides $p(X) = X^e - u$. Also, $(p(X), p'(X))D[X] = D[X]$, since e and u are units in D, so $(f(X), f'(X))D[X] = D[X]$, so E is locally unramified over D, hence M_iE is a radical ideal for $i = 1,\dots,n$. Further, E is a separable extension of D, since e is a unit in D, so the conductor of the integral closure E' of E in E contains $f'(u)$, by [21, (12.1.1)], and $f'(u)$ is a unit in E, since $(f(X), f'(X))D[X] = D[X]$, hence E is integrally closed. Therefore E is an integrally closed Noetherian domain of altitude one, so E is a Dedekind domain, by [22, Theorem 13, p. 275], so E is a semi-local PID by [22, Theorem 16, p. 278], since E has only finitely many maximal ideals. $\square$

The following corollary of Lemma 3.1 connects the Rees integers of I and the Rees integers of $IR[x]$ in the case where the positive integer m is a unit in R and x is a "formal" mth root of an element in R that is a unit in all Rees valuation rings of I.

Corollary 3.2. *With the notation of* (2.3)*, let* $P_1, \dots, P_n$ *be the centers in* R *of the Rees valuation rings of* I*, let* c *be a regular element in* $U = R \setminus (P_1 \cup \cdots \cup P_n)$*, let* m *be a positive integer, and let* $B_0 = R[x]$*, where* $x = X + (X^m - c)R[X]$*. If* m *is a unit in* R*, then the Rees integers of* I *and* IB_0 *are the same in the sense that a positive integer* e *is a Rees integer of* I *if and only if it is a Rees integer of* IB_0*. If* R *is an integral domain, then the Rees integers of* I*,* IB_0*, and* $(IB_0 + z)/z$ *are the same*[2] *for all minimal prime ideals* z *in* B_0*.*

Proof. For a sufficiently large positive integer n, there exists a regular element $b \in I^n$ such that $bV = I^n V$ for each $V \in \operatorname{Rees} I = \operatorname{Rees} I^n$, by [12, Lemma 3.1]. By (2.1.7), the Rees valuation rings of I correspond to the localizations at the height one associated prime ideals $p_1, \dots, p_r$ of $bR[I^n/b]'$, where $R[I^n/b]'$ is the integral closure of $R[I^n/b]$ in its total quotient ring. Therefore they correspond to the height one maximal ideals in $D = R[I^n/b]'_S$, where $S = R[I^n/b]' \setminus (p_1 \cup \cdots \cup p_r)$. Similarly, the Rees valuation rings of IB_0 correspond to the localizations at the height one associated prime ideals $q_1, \dots, q_{r'}$ of $bB_0[I^n/b]'$, so they correspond to the height one maximal ideals in $E = B_0[I^n/b]'_{S'}$, where $S' = B_0[I^n/b]' \setminus (q_1 \cup \cdots \cup q_{r'})$. Since the image of c in D is a unit, since minimal prime ideals in B_0 lie over minimal prime ideals in R (since B_0 is a free integral extension of R), and since, for each minimal prime ideal z in E, the factor rings $D/(z \cap D) \subseteq E/z \cong (D/(z \cap D))[(c + (z \cap D))^{1/m}]$ satisfy the hypothesis of Lemma 3.1 (with $n = r$, $g = 1$, and $e_1 = e = m$), the conclusions follow from Lemma 3.1 (and (2.1.8)). □

Lemma 3.3 is, in a certain sense, a dual of Lemma 3.1.

Lemma 3.3. *Let* $(D, M_1, \dots, M_n)$ *be a semi-local PID, fix* $M = M_1$*, let* π *be a prime element in* D *such that* $M = \pi D$*, let* m *be a positive integer, let* $E = D[\pi^{\frac{1}{m}}]$*, and let* $N = \pi^{\frac{1}{m}} E$*, where* $\pi^{\frac{1}{m}}$ *is an* m*th root of* π *in an algebraic closure of the quotient field of* D*. Then* E *is a finite integral extension domain of* D*, the ideal* $N = \pi^{\frac{1}{m}} E$ *is a maximal ideal in* E*, and* $ME = N^m$*, so* ME *is* N*-primary. If* m *is a unit in* D_S*, where* $S = D \setminus (M_2 \cup \cdots \cup M_n)$*, then* E *is a semi-local PID and* $M_i E$ *is a radical ideal for* $i = 2, \dots, n$*.*

Proof. It is clear that E is a finite integral extension domain of D. Let Q be a maximal ideal in E that lies over M. Then $(M, \pi^{\frac{1}{m}})E \subseteq Q$, since $\pi \in M$. Also, $(M, \pi^{\frac{1}{m}})E$ is a maximal ideal in E, since $E/(M, \pi^{\frac{1}{m}})E \cong D/M$ is a field. Further, $(M, \pi^{\frac{1}{m}})E$ is the principal maximal ideal $\pi^{\frac{1}{m}} E$, since $M = \pi D$. Therefore $Q = N = \pi^{\frac{1}{m}} E$ is the only maximal ideal in E that contains ME, so ME is a primary ideal and $ME = N^m$.

Now assume that m is a unit in D_S, where $S = D \setminus (M_2 \cup \cdots \cup M_n)$. Then, since $M = \pi D$, π is a unit in D_{M_i} for $i = 2, \dots, n$, so Lemma 3.1 implies that $M_i D_S[\pi^{\frac{1}{m}}]$ is a radical ideal for $i = 2, \dots, n$. Since $D_S[\pi^{\frac{1}{m}}]$ is a localization of E, and since $\pi^{\frac{1}{m}} E$ is the only prime ideal in E that lies over M, it follows that $M_i E$ is a radical ideal for $i = 2, \dots, n$. That E is a semi-local PID now follows by an argument as at the end of the proof of Lemma 3.1. □

[2] Again in the sense that a positive integer is a Rees integer for one of these ideals if and only if it is for the others. It often happens that IB_0 and even $(IB_0 + z)/z$ have more Rees valuation rings than I and therefore also more Rees integers (the same integers as for I, but some of these integers possibly occurring more frequently).

Corollary 3.4. *Let $(D, M_1, \ldots, M_n)$ be a semi-local PID, for $i = 1, \ldots, n$ let π_i be a prime element in E that generates M_i, let $e_1, \ldots, e_n$ be positive integers that are units in D, for $i = 1, \ldots, n$ let $\pi_i^{\frac{1}{e_i}}$ be an e_ith root of π_i in an algebraic closure of the quotient field of D, and let $E = D[\pi_1^{\frac{1}{e_1}}, \ldots, \pi_n^{\frac{1}{e_n}}]$. Then E is a semi-local PID such that, for each M_i and for each maximal ideal N in E that lies over M_i, $N^{e_i} E_N = M_i E_N$.*

Proof. Lemma 3.3 implies that E is a semi-local PID. For $i = 1, \ldots, n$ let $A_i = D[\pi_i^{\frac{1}{e_i}}]$, so $E = A_i[\pi_1^{\frac{1}{e_1}}, \ldots, \pi_{i-1}^{\frac{1}{e_{i-1}}}, \pi_{i+1}^{\frac{1}{e_{i+1}}}, \ldots, \pi_n^{\frac{1}{e_n}}]$. Since π_i is a unit in D_{S_i}, where $S_i = D \setminus (M_1 \cup \cdots \cup M_{i-1} \cup M_{i+1} \cup \cdots \cup M_n)$, Lemma 3.1 implies that $M_j A_i$ is a radical ideal, for $j = 1, \ldots, i-1, i+1, \ldots, n$. And Lemma 3.3 implies that $M_i A_i = Q_i^{e_i}$, where $Q_i = \pi_i^{\frac{1}{e_i}} A_i$ is a maximal ideal of A_i. Since $\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n$ are units in $A_{i\,Q_i}$, Lemma 3.1 implies that $Q_i E$ is a radical ideal. The conclusion clearly follows from this. □

A reason for using a common multiple, rather than the least common multiple, of the Rees integers of I in Proposition 3.5 is noted immediately following Theorem 3.7.3. Also, a multiple of the least common multiple is needed in Corollaries 3.15 and 3.18.

Proposition 3.5. *Let b be a nonzero element in the Jacobson radical of a semi-local PID $(D, M_1, \ldots, M_n)$, for $i = 1, \ldots, n$ let π_i be a prime element in D that generates M_i, and let $b = \mu\pi_1^{e_1} \cdots \pi_n^{e_n}$, where μ is a unit in D and $e_1, \ldots, e_n$ are positive integers. Let m be a common multiple of $e_1, \ldots, e_n$ which is a unit in D, and let $b^{\frac{1}{m}}$ be an mth root of b in an algebraic closure of the quotient field of D. Then the integral closure E' of $E = D[b^{\frac{1}{m}}]$ in its quotient field is a semi-local PID and an integral extension domain of D, and $b^{\frac{1}{m}} E'$ is the Jacobson radical of E'.*

Proof. Since $E = D[b^{\frac{1}{m}}]$ is a simple integral extension domain of D, and since D is a semi-local PID, it follows that E' is an integral extension domain of D of altitude one with only finitely many maximal ideals, by [14, (33.10)]. Since E' is integrally closed, it is a semi-local PID.

Fix $i \in \{1, \ldots, n\}$ and let $V = D_{M_i}$ and $M = M_i V$. Let $f_i = \frac{m}{e_i}$. It follows from Lemma 3.3 that $U = V[\pi_i^{\frac{e_i}{m}}]$ is a valuation domain whose maximal ideal is $P = \pi_i^{\frac{1}{f_i}} U$ and that $MU = P^{f_i}$. Let $G = V[\pi_i^{\frac{e_i}{m}}, \nu_i^{\frac{1}{m}}]$, where $\nu_i = \mu\pi_1^{e_1} \cdots \pi_{i-1}^{e_{i-1}} \pi_{i+1}^{e_{i+1}} \cdots \pi_n^{e_n}$ is a unit in V. Since m is a unit in $D \subseteq V$, Lemmas 3.1 and 3.3 imply that G is a semi-local PID. Since $G = U[\nu_i^{\frac{1}{m}}]$, Lemma 3.1 implies that PG is a radical ideal. It follows that each primary component of $M_i G = (P^{f_i} U)G$ is the f_ith power of some maximal ideal in G that lies over M_i. Since $E = D[b^{\frac{1}{m}}] \subseteq G$ (so $E' \subseteq G$ and G is integral over E'_{D-M_i} (in fact, it is readily checked that $G = E'_{D-M_i}[\nu_i^{\frac{1}{m}}, \pi_i^{\frac{1}{f_i}}] = V[b^{\frac{1}{m}}][\nu_i^{\frac{1}{m}}, \pi_i^{\frac{1}{f_i}}]$)), it follows that $M_i E'$ cannot be contained in the $(f_i + 1)$th power of any maximal ideal in E'.

On the other hand, since b is in the Jacobson radical of D, it follows that if N is a maximal ideal in E' that lies over M_i, and if v is the valuation of E'_N, then $v(\pi_i) \geqslant 1$, so

$1 \leqslant v(b^{\frac{1}{m}}) = v((\pi_i^{e_i})^{\frac{1}{m}}) = \frac{e_i}{m}v(\pi_i)$, so $v(\pi_i) \geqslant \frac{m}{e_i} = f_i$. Therefore $M_i E'_N \subseteq N^{f_i} E'_N$. Since $M_i E'$ is not contained in the $(f_i + 1)$th power of N, we have $N^{f_i} E'_N = M_i E'_N$.

Finally, since $b \in \pi_i^{e_i} D$ and since $e_i f_i = m$, it follows from what was just shown that, for each maximal ideal N in E', $bE'_N = N^m E'_N$. Therefore $b^{\frac{1}{m}} E'$ is the Jacobson radical of E'. □

We show in Corollary 3.10 that a result analogous to Proposition 3.5 holds for all regular principal ideals in an arbitrary Noetherian ring.

Remark 3.6. If the greatest common divisor of the Rees integers $e_1, \ldots, e_n$ of I is equal to one, then I is projectively full, by [1, (4.10)].[3] If there exists an ideal $K \in \mathbf{P}(I)$ whose Rees integers have greatest common divisor equal to one, then K and $\mathbf{P}(I)$ are projectively full. Since the ordered sets of Rees integers of I and K are proportional, by [11, Proposition 2.10], and since $\mathbf{P}(I) = \mathbf{P}(K)$ is linearly ordered by inclusion, by [11, Corollary 2.4], necessarily K is the largest ideal in $\mathbf{P}(I)$.

Theorem 3.7. *Let I be a regular proper ideal in a Noetherian ring R and let $b_1, \ldots, b_g$ be regular elements that generate I. Let $(V_1, N_1), \ldots, (V_n, N_n)$ be the Rees valuation rings of I. Assume*[4] *that $b_i V_j = I V_j$ $(= N_j^{e_j}$, say) for $i = 1, \ldots, g$ and $j = 1, \ldots, n$, and that the least common multiple e of $e_1, \ldots, e_n$ is a unit in R. Let m be a positive multiple of e that is a unit in R. With the notation of* (2.3) *let $A_m = R[x_1, \ldots, x_g]$, let $J_m = (x_1, \ldots, x_g)A_m$, and let $A = A_m$ and $J = J_m$. Then*:

(3.7.1) *A is a finite free integral extension ring of R, $J_a = \mathrm{Rad}(IA)$ is a projectively full radical ideal that is projectively equivalent to IA, $(IA)_a = (J^m)_a$, and for each Rees valuation ring U of J and for $i = 1, \ldots, g$ it holds that $x_i U = JU$ is the maximal ideal of U, so the Rees integers of J are all equal to one.*

(3.7.2) *Assume that R is a Noetherian domain, let z be a minimal prime ideal in A, and let $\overline{A} = A/z = R[b_1^{\frac{1}{m}}, \ldots, b_g^{\frac{1}{m}}]$ and $\overline{J} = J/z = (b_1^{\frac{1}{m}}, \ldots, b_g^{\frac{1}{m}})\overline{A}$ (see* (2.3)*). Then $\overline{A}$ is a Noetherian domain that is a finite integral extension ring of R, $\mathrm{Rad}(I\overline{A}) = \overline{J}_a$ is a projectively full radical ideal that is projectively equivalent to $I\overline{A}$, $(I\overline{A})_a = (\overline{J}^m)_a$, and the Rees integers of $\overline{J}$ are all equal to one.*

Proof. For (3.7.1), let $G = R[I/b_1]$, let $B_0 = R$ and $C_0 = G$, and for $i = 1, \ldots, g$ let $B_i = R[x_1, \ldots, x_i]$ and $C_i = G[x_1, \ldots, x_i]$, so $B_i = B_{i-1}[x_i]$ and $C_i = C_{i-1}[x_i]$, where x_i is the image of X_i in the residue class ring $B_{i-1}[X_i]/((X_i^m - b_i)B_{i-1}[X_i])$. Then it is clear that B_i (respectively, C_i) is a finite free integral extension ring of B_{i-1} (respectively, C_{i-1}), so $A = B_g$ (respectively, C_g) is a finite free integral extension ring of $B_0 = R$ (respectively, $C_0 = G$). Also, it follows from [15, Definition, p. 213 and Proposition 2.13] that, for each associated prime ideal $p_{i,j}$ of $b_1 C'_i$ (where C'_i is the integral closure of C_i in its total quotient ring), there exists a unique minimal prime ideal $z_{i,j} \subset b_1 C'_{i\,p_{i,j}}$ such that $W_{i,j} = C'_{i\,p_{i,j}}/z_{i,j}$ is a discrete valuation domain (possibly $z_{i,h} = z_{i,j}$ for some $h \neq j$). Therefore, since $C_i = B_i[I/b_1]$, it follows from

[3] The converse is false, by [11, Example 3.4]. Indeed, it is observed in [4, Example 3.1] that if (R, M) is a regular local ring of altitude two with $M = (x, y)R$, then for each integer $e > 1$ the ideal $I = (x, y^e)R$ is projectively full and the gcd of the Rees integers of I is e.

[4] Concerning the hypothesis "$b_i V_j = I V_j$ for $i = 1, \ldots, g$ and $j = 1, \ldots, n$," there exists such a basis for I if either R contains an infinite field (Lemma 3.19), or if R is a local ring with an infinite residue field (Remark 3.20).

the definition of Rees valuation rings that the Rees valuation rings of IB_i are these valuation rings $W_{i,j}$ (they are *all* the Rees valuation rings of IB_i, since $b_1V_j = IV_j$ for $j = 1, \dots, n$ (by hypothesis) and C_i' is integral over $C_0' = G'$ imply that $b_1W_{i,j} = IW_{i,j}$ for all $W_{i,j}$).

For $i = 0, \dots, g$ let $E_i = C'_{i\,S_i}$, where S_i is the complement in C_i' of the union of all the associated prime ideals of b_1C_i', and let z be a minimal prime ideal in E_g, so for $i = 0, \dots, g-1$, $z_i = z \cap E_i$, $z \cap C_i$, $z \cap B_i$, and $z \cap R$ are minimal prime ideals, and E_i/z_i and E_g/z are semi-local PIDs (by [15, Corollary 2.12(2) and (2.7)]). Also, it follows from the preceding paragraph that E_i/z_i is the integral closure of $(E_{i-1}/z_{i-1})[(b_i + (z_{i-1}))^{\frac{1}{m}}]$ in the quotient field of E_i/z_i.

If $i = 1$, then, the localizations $(E_0)_Q$ of E_0 (as Q runs over all the maximal ideals $Q_1, \dots, Q_n$ of E_0) are the Rees valuation rings of I, so it follows from the hypothesis that, for $h = 1, \dots, g$, $b_hE_0 = \pi_1^{e_1} \cdots \pi_n^{e_n} E_0$, where $\pi_j \in Q_j \setminus (Q_1 \cup \cdots \cup Q_{j-1} \cup Q_{j+1} \cup \cdots \cup Q_n)$ such that $\pi_j(E_0)_{Q_j} = Q_j(E_0)_{Q_j}$. Therefore it follows from Proposition 3.5 that

$$(3.7.3) \quad (b_1 + z_0)^{\frac{1}{m}}(E_1/z_1) \quad \text{is the Jacobson radical of } E_1/z_1,$$

so since E_1 is integral over E_0 and $b_hE_0 = b_1E_0$ for $h = 1, \dots, g$, it follows from (3.7.3) that

$$(3.7.4) \quad (b_h + z_0)^{\frac{1}{m}}(E_1/z_1) \quad \text{is the Jacobson radical of } E_1/z_1, \text{ for } h = 1, \dots, g.$$

(The least common multiple of the Rees integers of I corresponding to the Rees valuation rings of I that contain $R/(z \cap R)$ is a factor of e, but may fail to be equal to e; this is a reason for not restricting to the least common multiple of $e_1, \dots, e_n$ in Proposition 3.5.)

Now let $i \in \{2, \dots, g\}$ and assume that $(b_h + z_{i-1})(E_{i-1}/z_{i-1}) = Z_{i-1}^m$ for $h = 1, \dots, g$, where Z_{i-1} is the Jacobson radical of E_{i-1}/z_{i-1}. (This holds for $i = 2$, by (3.7.4).) Then it follows from Proposition 3.5 that

$$(3.7.5) \quad (b_i + z_{i-1})^{\frac{1}{m}}(E_i/z_i) \quad \text{is the Jacobson radical of } E_i/z_i,$$

so since E_i is integral over E_{i-1} and $b_hE_{i-1} = b_iE_{i-1}$ for $h = 1, \dots, g$, it follows from (3.7.5) that

$$(3.7.6) \quad (b_h + z_{i-1})^{\frac{1}{m}}(E_i/z_i) \quad \text{is the Jacobson radical of } E_i/z_i, \text{ for } h = 1, \dots, g.$$

The localizations of E_i/z_i at its maximal ideals are (some of the) Rees valuation rings of IB_i (by the definition of E_i and by what was noted at the end of the first paragraph of this proof). Therefore it follows from (3.7.6) that the Rees integers of IB_i with respect to these valuation rings are all equal to m and that, for $h = 1, \dots, g$, $x_hW_{i,j} = N_{i,j}$ for all Rees valuation rings $(W_{i,j}, N_{i,j})$ of IB_i. Since this holds for each minimal prime ideal z in E_g, it follows (since $A = A_g$) that if (U, Q) is a Rees valuation ring of IA, then $IU = Q^m$ and $x_hU = Q$ for $h = 1, \dots, g$. Since $J = (x_1, \dots, x_g)A$, it follows from Remark 3.6 that J is projectively full, and since $x_h^m = b_h$ for $h = 1, \dots, g$, it follows that $(J^m)_a = (IA)_a$, so J is projectively equivalent to IA. Since the Rees integers of J are all equal to one, [3, (4.1.2)] shows that $uA[u, tJ]'$ is a radical ideal, where $A[u, tJ]'$ is the integral closure of the Rees ring $A[u, tJ]$ (of A with respect to J) in its total quotient ring. Since $uA[u, tJ]' = \mathrm{Rad}(uA[u, tJ]')$, since $uA[u, tJ]' \cap A = J_a$, and since $\mathrm{Rad}(uA[u, tJ]') \cap A = \mathrm{Rad}(J)$, it follows that J_a is a radical ideal.

For (3.7.2), assume that R is a Noetherian domain and for $i = 1, \ldots, g$ fix an mth root $b_i^{\frac{1}{m}}$ of b_i in an algebraic closure of the quotient field of R. Then the proof for this case is essentially the same as for (3.7.1), but replace $B_i = R[x_1, \ldots, x_i]$ with $\overline{B}_i = R[b_1^{\frac{1}{m}}, \ldots, b_i^{\frac{1}{m}}]$. $\square$

It is shown at the end of the last paragraph of the proof of Theorem 3.7.1 that J_a is a radical ideal. An alternate proof of this is: $J_a = (IA)_{(\frac{1}{m})}$ (since $(J^m)_a = (IA)_a$), so $J_a \cap R = I_{(\frac{1}{m})}$, by Theorem 2.6.2, and $A/J_a \cong R/I_{(\frac{1}{m})}$ is a reduced ring, by Theorems 2.6.1 and 2.6.3, hence J_a is a radical ideal.

Remark 3.8.
(3.8.1) The ring A of Theorem 3.7 is the same as the ring A_m of Theorem 2.6, but with the additional assumption concerning the basis elements $b_1, \ldots, b_g$ of I and the assumption that the integer m (of Theorem 2.6) is a unit, and the ideal J of Theorem 3.7 is the same as the ideal J_m of Theorem 2.6. Therefore it follows from Theorems 2.6.2 and 2.6.4 that if J is as in Theorem 3.7, then, for all positive integers k, $(J^k)_a \cap R = I_{(\frac{k}{m})}$, and if R is an integral domain, then $(((J + z)/z)^k)_a \cap R = I_{(\frac{k}{m})}$. Also, since m is a multiple of the integer $d(I)$ (of (2.1.5)), $\mathbf{P}(I) \subseteq \mathcal{F}_m = \{I_{(\frac{k}{m})}\}_{k \geqslant 0}$ and $\mathcal{F}_{d(I)} \setminus \mathbf{P}(I)$ is a finite set, by Proposition 2.5.4, but if $m \neq d(I)$, then $\mathcal{F}_m \setminus \mathbf{P}(I)$ may be infinite; a specific example is given in Example 3.22 below.
(3.8.2) If I is a prime ideal (respectively, a radical ideal), and if J is the ideal $(x_1, \ldots, x_g)A$ of Theorem 3.7.1, then the proof of Theorem 2.6.3 shows that $J \cap R = I$ and $A/J \cong R/I$. Hence J $(= J_a)$ is a prime ideal (respectively, a radical ideal having the same number of minimal associated prime ideals as I).
(3.8.3) In Theorem 3.7, the hypothesis "$b_i V_j = I V_j$ for $i = 1, \ldots, g$ and $j = 1, \ldots, n$" can be replaced by an analogous hypothesis on a power I^k of I.

Proof. For (3.8.3), the ideals I and I^k have the same Rees valuation rings and the Rees integers of I^k are ke_i for $i = 1, \ldots, n$. Also, if the least common multiple of the Rees integers $e_1, \ldots, e_n$ of I is a unit in R, then there exist infinitely many large integers k such that the least common multiple of the Rees integers $ke_1, \ldots, ke_n$ of I^k is a unit in R, so Theorem 3.7 can be applied to I^k in place of I. Finally, since I and I^k are projectively equivalent, it follows that the ring A and ideal J for I^k also works for I. $\square$

In Theorem 3.7, it would be nice if the hypothesis "$b_i V_j = I V_j$ for all i, j" could be omitted. As noted in the footnote of 3.7, there are cases when this hypothesis automatically holds, and Remark 3.8.3 shows this hypothesis can be replaced with the analogous statement for some power I^k of I. Because of this, we would be interested in knowing the answer to the following:

Question 3.9. Let I be a regular ideal in a Noetherian ring R. Does there always exist a positive integer k such that there exist regular elements $b_1, \ldots, b_g$ that generate I^k and have the property that $b_i V_j = I^k V_j$ for $i = 1, \ldots, g$ and $j = 1, \ldots, n$? Does this hold for all sufficiently large integers k?

Corollary 3.10 is an analog to Proposition 3.5 for regular principal ideals in an arbitrary Noetherian ring.

Corollary 3.10. *Let b be a regular nonunit in a Noetherian ring R, let R' be the integral closure of R in its total quotient ring, let $bR' = p_1^{(e_1)} \cap \cdots \cap p_n^{(e_n)}$, where $p_i^{(e_i)}$ is the e_ith symbolic power of the height one prime ideal p_i of R' ($i = 1, \ldots, n$), and let m be a common multiple of $e_1, \ldots, e_n$. Assume that m is a unit in R and let $A = R[x] = R[X]/((X^m - b)R[X])$ and $J = xA$. Then*:

(3.10.1) *A is a finite free integral extension ring of R, J_a is a projectively full radical ideal that is projectively equivalent to bA, and the Rees integers of J are all equal to one.*

(3.10.2) *If R is a Noetherian domain, then for each minimal prime ideal z in A it holds that $((J + z)/z)_a$ is a projectively full radical ideal that is projectively equivalent to $b(A/z)$, and the Rees integers of $b^{\frac{1}{m}}(A/z) = (J + z)/z$ are all equal to one.*

Proof. It follows from Definition 2.1.8 that $e_1, \ldots, e_n$ are the Rees integers of bR, so this is the case $g = 1$ of Theorem 3.7. □

We use the following definition in additional remarks about Theorem 3.7.

Definition 3.11. Let I be a regular proper ideal in a Noetherian ring R and let $\hat{A}^*(I)$ denote the set of *asymptotic prime divisors* of I, thus $\hat{A}^*(I) = \{P \in \mathrm{Spec}(R) \mid P \in \mathrm{Ass}(R/(I^k)_a)$ for some positive integer $k\}$.

Remark 3.12. It is clear that every minimal associated prime ideal of I is in $\hat{A}^*(I)$. Also, it is shown in [10, (3.18)] that $\hat{A}^*(I)$ is the set of centers in R of the Rees valuation rings of I (see also [1, (2.8)]), so $\hat{A}^*(I)$ is a finite set.

Remark 3.13.

(3.13.1) In Theorem 3.7.1, the ideals J and J_a may have embedded asymptotic prime divisors, even though their Rees integers are all equal to one. The analogous statement holds for $\overline{J}$ and $\overline{J}_a$ in Theorem 3.7.2. For example, let (R, M) be the local domain of [14, Example 2, pp. 203–205] in the case $r = 1$ and $m = 0$, and let $I = (x^2 - x)R$. (Much the same example appears in [9, pp. 87–88] and in [23, pp. 327–329].) In this example, (R, M) is a local domain of altitude two and its integral closure R' has the following properties: $R' = R + bR$ for each $b \in R' \setminus R$; R' has exactly two maximal ideals $M_1 = xR'$ and $M_2 = (x - 1, z_1)R'$; R'_{M_1} and R'_{M_2} are regular local domains of altitude one and two, respectively; and, $M = M_1 \cap M_2$. It follows that the Rees valuation rings of $I = (x^2 - x)R$ are $R'_{xR'}$ and $R'_{(x-1)R'}$, so the Rees integers of I are both equal to one; therefore the least common multiple e of the Rees integers of I is one, so in Theorem 3.7 we may take $m = 1$, $A = R$, and $J = I$. Then $J_a = I_a = (x^2 - x, x^3 - x^2)R$, so $R[J_a/(x^2 - x)] = R[x] = R'$, so it is readily checked that J_a is a height one prime ideal, but it follows from Remark 3.12 that $\hat{A}^*(J) = \{J_a, M\}$.

(3.13.2) Let J be as in Theorem 3.7.1 (so the Rees integers of J are all equal to one), let B be a Noetherian integral extension ring of A contained in the total quotient ring of A, and let H be an ideal in B such that $JB \subseteq H \subseteq P$, where P is an arbitrary prime ideal in $\hat{A}^*(JB)$. Then H is projectively full and, in fact, has P as an asymptotic prime divisor with corresponding Rees integer equal to one. (This follows from [3, (2.9), (2.10), and (4.2.3)].)

(3.13.3) Let J be as in Theorem 3.7.1 and assume that J has no embedded asymptotic prime divisors. Then for all integers k it holds that $(J^k)_a = \bigcap\{(P^k)_a A_P \cap A \mid P \in \hat{A}^*(J)\}$ (since each Rees integer of J is equal to one).

Corollary 3.15 extends Theorem 3.7 to certain finite sets of ideals. In the proof of Corollary 3.15 we use the following remarks.

Remark 3.14.
(3.14.1) It is shown in [17, (2.5.2)] that if $b_1, \dots, b_g$ are regular elements in a Noetherian ring R and if $e_1, \dots, e_g$ are positive integers, then $\hat{A}^*((b_1, \dots, b_g)R) = \hat{A}^*((b_1^{e_1}, \dots, b_g^{e_g})R)$.
(3.14.2) It is shown in [16, (3.3.4)] that if A is an integral extension ring of a Noetherian ring R such that minimal prime ideals in A lie over minimal prime ideals in R, and if I is a regular proper ideal in R, then $\hat{A}^*(I) = \{P \cap R \mid P \in \hat{A}^*(IA)\}$.

Corollary 3.15. *Let $I_1, \dots, I_h$ be regular proper ideals in a Noetherian ring R. For $i = 1, \dots, h$ let $b_{i,1}, \dots, b_{i,g_i}$ be regular elements in R that generate I_i, assume that, for $i \neq i'$ in $\{1, \dots, h\}$, no $b_{i,j}$ $(j = 1, \dots, g_i)$ is in any $p \in \hat{A}^*(I_{i'})$ (so there are no containment relations among the ideals in $\hat{A}^*(I_i)$ and the ideals in $\hat{A}^*(I_{i'})$), and also assume that $b_{i,j}V_{i,k} = I_iV_{i,k}$ for $i = 1, \dots, g$, for $j = 1, \dots, g_i$, and for all Rees valuation rings $V_{i,k}$ of I_i. Let e_i be the least common multiple of the Rees integers of I_i, let m be a common multiple of $e_1, \dots, e_h$, let $B = R[\{x_{i,j} \mid i = 1, \dots, h$ and $j = 1, \dots, g_i\}]$, where, for $i = 1, \dots, h$ and $j = 1, \dots, g_i$, $x_{i,j}$ is a "formal" mth root of $b_{i,j}$ (see (2.3)), and let $H_i = (x_{i,1}, \dots, x_{i,g_i})B$ $(i = 1, \dots, h)$. If m is a unit in R, then*:
(3.15.1) *For $i = 1, \dots, h$ it holds that*: $(I_iB)_a = (H_i^m)_a$ *(so I_iB is projectively equivalent to H_i)*; $(H_i)_a = \mathrm{Rad}(H_i)$ *is projectively full*; *and, the Rees integers of H_i are all equal to one.*
(3.15.2) *Assume that R is an integral domain and let z be a minimal prime ideal in B. Then for $i = 1, \dots, h$ it holds that*: $(I_i(B/z))_a = (((H_i + z)/z)^m)_a$ *(so $I_i(B/z)$ is projectively equivalent to $(H_i + z)/z$)*; $((H_i + z)/z)_a = \mathrm{Rad}((H_i + z)/z)$ *is projectively full*; *and, the Rees integers of $(H_i + z)/z$ are all equal to one.*

Proof. For (3.15.1), it follows from Remark 2.7.1 that $(I_iB)_a = (H_i^m)_a$, so I_iB is projectively equivalent to H_i. To see that the Rees integers of the ideals H_i are all equal to one, fix $i \in \{1, \dots, h\}$, and let $R_i = R[x_{i,1}, \dots, x_{i,g_i}]$. Since m is a common multiple of $e_1, \dots, e_h$ (where e_i is the least common multiple of the Rees integers of I_i), it follows from Theorem 3.7.1 that $(I_iR_i)_a = (((x_{i,1}, \dots, x_{i,g_i})R_i)^m)_a$ (so I_iR_i is projectively equivalent to $(x_{i,1}, \dots, x_{i,g_i})R_i$), $((x_{i,1}, \dots, x_{i,g_i})R_i)_a = \mathrm{Rad}((x_{i,1}, \dots, x_{i,g_i})R_i)$ is projectively full; and, the Rees integers of $(x_{i,1}, \dots, x_{i,g_i})R_i$ are all equal to one.

Let $S = \{x_{f,k} \mid f \in \{1, \dots, i-1, i+1, \dots, g\}$ and $k \in \{1, \dots, g_f\}\}$ (so $B = R_i[S]$), let $S_0 \subseteq S$, let $C_1 = R_i[S_0]$, and assume that the Rees integers of $(x_{i,1}, \dots, x_{i,g_i})C_1$ are all equal to one. (Note that this holds for $S_0 = \emptyset$, by the preceding paragraph.) Let $x \in S - S_0$ (say x is a "formal" mth root of $b_{f,k}$), and let $C_2 = C_1[x]$. Now $b_{f,k} \notin \bigcup\{p \mid p \in \hat{A}^*(I_i)\}$, by hypothesis, so it follows from Remarks 3.14.1 and 3.14.2 that $b_{f,k} \notin \bigcup\{P \mid P \in \hat{A}^*((x_{i,1}, \dots, x_{i,g_i})C_1)\}$. Therefore it follows from Corollary 3.2 (together with Remark 3.14.1) that the Rees integers of $(x_{i,1}, \dots, x_{i,g_i})C_2$ are all equal to one. By iterating this, it follows that the Rees integers of H_i are all equal to one. It therefore follows as at the end of the last paragraph of the proof of Theorem 3.7.1 that $(H_i)_a = \mathrm{Rad}(H_i)$ is projectively full.

The proof of (3.15.2) is similar to the proof of (3.15.1), since (3.7.2) shows that if R is a Noetherian domain, then the Rees integers of $((x_{i,1}, \dots, x_{i,g_i})R_i + z)/z$ are all equal to one, and then iterated use of Remarks 3.14.1, 3.14.2, and Corollary 3.2 shows that the Rees integers of $(H_i + z)/z$ are all equal to one. $\square$

Corollary 3.18 combines Corollaries 3.10 and 3.15. For this corollary we use the following definition and remarks.

Definition 3.16. Regular elements $b_1, \dots, b_g$ in a Noetherian ring R are an *asymptotic sequence in* R in case $(b_1, \dots, b_g)R \neq R$ and $b_{i+1} \notin \bigcup\{P \mid P \in \hat{A}^*(b_1, \dots, b_i)R\}$ for $i = 1, \dots, g-1$. They are a *permutable asymptotic sequence in* R in case each permutation of them is an asymptotic sequence in R.

Remark 3.17.
(3.17.1) If $b_1, \dots, b_g$ are an asymptotic sequence contained in the Jacobson radical of a Noetherian ring R, then they are a permutable asymptotic sequence, by [10, (5.5)].
(3.17.2) If I is an ideal generated by the asymptotic sequence $b_1, \dots, b_g$, if I is contained in the Jacobson radical of R, and if $c_1, \dots, c_g$ are any g elements that generate I, then $c_1, \dots, c_g$ are an asymptotic sequence in R, by [10, (5.19)].
(3.17.3) If $b_1, \dots, b_g$ are an R-sequence, then they are an asymptotic sequence in R, by [10, (5.13)].

Concerning the hypothesis in Corollary 3.18, if R is a local ring with an infinite residue field, and if $c_1, \dots, c_g$ is an asymptotic sequence in R, then Remark 3.20 shows that there exist elements $b_1, \dots, b_g$ in R such that $(b_1, \dots, b_g)R = (c_1, \dots, c_g)R$ and $b_i V = (c_1, \dots, c_g)V$ for $i = 1, \dots, g$ and for all Rees valuation rings V of $(c_1, \dots, c_g)R$, and it follows from Remarks 3.17.1 and 3.17.2 that $b_1, \dots, b_g$ are a permutable asymptotic sequence in R. It follows similarly by using Lemma 3.19 in place of Remark 3.20 that the analogous statement holds if R is a Noetherian ring that contains an infinite field.

Corollary 3.18. *Let $b_1, \dots, b_g$ be a permutable asymptotic sequence in a Noetherian ring R and let $I = (b_1, \dots, b_g)R$. For $i = 1, \dots, g$ let e_i be the least common multiple of the Rees integers of $b_i R$, let e be the least common multiple of the Rees integers of I, and let m be a common multiple of $e_1, \dots, e_g, e$. Assume that m is a unit in R and that $b_i V = IV$ for $i = 1, \dots, g$ and for all Rees valuation rings V of I. With the notation of* (2.3) *let $A_m = R[x_1, \dots, x_g]$ and $J_m = (x_1, \dots, x_g)A$, and let $A = A_m$ and $J = J_m$. Also, for $i = 1, \dots, g$ let $H_i = x_i A$. Then*:
(3.18.1) *J_a (respectively, $(H_1)_a, \dots, (H_g)_a$) is a projectively full radical ideal that is projectively equivalent to IA (respectively, $b_1 A, \dots, b_g A$) and the Rees integers of J (respectively, $H_1, \dots, H_g$) are all equal to one.*
(3.18.2) *If R is an integral domain, then for each minimal prime ideal z in R, the analogous statements hold for the ideals $(J + z)/z, (H_1 + z)/z, \dots, (H_g + z)/z$.*

Proof. The statements concerning J_a and J follow immediately from Theorem 3.7, and the statements concerning the ideals $(H_i)_a$ and H_i $(i = 1, \dots, g)$ follow immediately from Corollaries 3.10 and 3.15 (since Definition 3.11, together with Remark 3.12, shows that the ideals $b_1 R, \dots, b_g R$ satisfy the hypothesis on the ideals $I_1, \dots, I_h$ in Corollary 3.15). □

We use Lemma 3.19 in the proof of Corollary 3.21.

Lemma 3.19. *Assume that R contains an infinite field F. Then there exist regular elements $b_1, \dots, b_g$ in R that are a basis of I such that $b_i V = IV$ for all Rees valuation rings V of I and for $i = 1, \dots, g$.*

Proof. Let $V_1, \dots, V_n$ be the Rees valuation rings of I, let v_i be the valuation of V_i, and let P_i be the center in R of V_i (possibly $P_i = P_j$ for some $i \neq j \in \{1, \dots, n\}$). It is readily checked that $S_i = \{b \in I \mid v_i(b) > v_i(I)\}$ is an ideal in R that contains $P_i I$ and is properly contained in I. Let $z_1, \dots, z_m$ be the associated prime ideals of (0) in R. Then $I \nsubseteq z_j$ for $j = 1, \dots, m$, since I is a regular ideal.

Now R is an F-vector space and the ideals of R are F-subspaces of R. Therefore $I, S_1, \dots S_n, z_1, \dots, z_m$ are F-subspaces of R such that I is not contained in any of the others. Since F is infinite there exists a vector space basis of I over F consisting of elements not in $S_1 \cup \dots \cup S_n \cup z_1 \dots \cup z_m$. Since R is Noetherian, it follows that there exist finitely many regular elements, say $b_1, \dots, b_g$, in R that are a basis of I such that $v_i(b_j) = v_i(I)$ for $i = 1, \dots, n$ and for $j = 1, \dots, g$. □

Remark 3.20. Concerning Lemma 3.19, by a similar result [3, (2.7.1)], if I is a regular proper ideal in a local ring (R, M), and if R/M is an infinite field, then there exist regular elements $b_1, \dots, b_g$ in R that are a basis of I such that $b_i V = IV$ for all Rees valuation rings V of I and for $i = 1, \dots, g$.

Corollary 3.21. *Assume that R is a Noetherian ring that contains the field $\mathbb{Q}$ of rational numbers. Then for each regular proper ideal I in R:*
(3.21.1) *There exists a finite free integral extension ring A of R that contains a projectively full radical ideal J^* that is projectively equivalent to IA and whose Rees integers are all equal to one.*
(3.21.2) *If R is an integral domain, then for each minimal prime ideal z in A the ideal $((J^* + z)/z)_a$ in A/z is a projectively full radical ideal that is projectively equivalent to $(IA + z)/z$, and the Rees integers of $(J^* + z)/z$ are all equal to one.*

Proof. This follows from Lemma 3.19 and Theorem 3.7.1. □

Concerning (1.1) in the Introduction, Example 3.22 shows that if the least common multiple of the Rees integers of I is not a unit in R, then the method used in the proof of Proposition 3.5 (respectively, Theorem 3.7) to obtain E' and $b^{\frac{1}{m}} E'$ (respectively, A and J) may fail to give such an extension domain A. However, it is shown in [5] that, at least for the integral domain D in Example 3.22, such an extension domain A does exist.

Example 3.22. Let $D = F[y]_S$, where F is a field of characteristic two, y is an indeterminate, and $S = F[y] \setminus (yF[y] \cup (y+1)F[y])$, let $I = bD$, where $b = y^2(y+1)$, let $J = xD[x]$, where $x = \sqrt{b}$, and let E be the integral closure of $D[x]$ in its quotient field. Then:
(3.22.1) D is a semi-local PID, $d(I) = 1$ (so I is projectively full), and I has the two Rees integers $e_1 = 1$ and $e_2 = 2$, so the least common multiple of the Rees integers of I is 2.
(3.22.2) E is a semi-local PID, $J_a = \mathrm{Rad}(J) = (x, y(y+1))D[x]$ is the Jacobson radical of $D[x]$, and the Rees integer of xE with respect to the extension of D_{yD} to $F(x)$ is 2, so not all the Rees integers of xE are equal to one and xE is not the Jacobson radical of E.
(3.22.3) For the filtration $\mathcal{F}_e = \mathcal{F}_2 = \{I_{(\frac{k}{2})}\}_{k \geqslant 0}$, $I_{(\frac{1}{2})} = y(y+1)D$ (the Jacobson radical of D), and for each positive integer k, the ideal $I_{(\frac{2k}{2})} = I^k = y^{2k}(y+1)^k D$ is projectively equivalent to I, but the ideal $I_{(\frac{2k+1}{2})} = y^{2k+1}(y+1)^{k+1} D$ is not projectively equivalent to I. Therefore $\mathcal{F}_e \setminus \mathbf{P}(I)$ is an infinite set.

Proof. For (3.22.1), it is clear that D is a semi-local PID. Also, I has the two Rees integers $e_1 = 1$ (for its Rees valuation ring D_{p_1}, where $p_1 = (y+1)D$) and $e_2 = 2$ (for its Rees valuation ring D_{p_2}, where $p_2 = yD$). Therefore the greatest common divisor of e_1, e_2 is one (so $d(I) = 1$ and I is projectively full, by Remark 3.6), and the least common multiple of e_1, e_2 is $e = 2$.

For (3.22.2), it is readily checked that $(X^2 - b)D[X]$ is a prime ideal, so $D[x]$ is a simple integral extension domain of D, so $D[x]$ is a semi-local (Noetherian) domain of altitude one, hence $E = D[x]'$ is a semi-local PID. Also, $(xD[x])_a = \mathrm{Rad}(xD[x])$, by Theorem 2.6.3.

To see that $\mathrm{Rad}(xD[x]) = (x, y(y+1))D[x]$ is the Jacobson radical of $D[x]$, note first that it follows from integral dependence that x is in the Jacobson radical of $D[x]$, since $x^2 = b = y^2(y+1)$ is in the Jacobson radical of D, so $(x, y(y+1))D[x]$ is contained in the Jacobson radical of $D[x]$. Also, $D[x]/((x, y(y+1))D[x]) = F \oplus F$ is the direct sum of two fields, so $(x, y(y+1))D[x]$ is a radical ideal, so $(x, y(y+1))D[x]$ is the Jacobson radical of $D[x]$. Further, $\sqrt{y+1} \in E \setminus D[x]$, and $y(y+1) = x\sqrt{y+1} \in D[x]$, so $y(y+1) \in xE \cap D[x] = (xD[x])_a$. It follows that $(xD[x])_a = (x, y(y+1))D[x]$ is the Jacobson radical of $D[x]$. (Note that $x\sqrt{y+1} \in (xD[x])_a \setminus xD[x]$.)

To complete the proof of (3.22.2), let v_y be the valuation of the extension V of D_{yD} to $F(x)$. Then $v_y(x) = v_y(\sqrt{b}) = v_y(\sqrt{y^2(y+1)}) = 1 = v_y(y)$, so $\frac{x}{y}$ is a unit in V. Also, $x^2 = b = y^2(y+1)$, so $(\frac{x}{y})^2 = y+1$, so $(\frac{x}{y}+1)^2 = y$, so $v_y(\frac{x}{y}+1) = 1/2$. By normalizing, it follows that the Rees integer of xE with respect to V is two, so xE has a Rees integer that is not equal to one and xE is not the Jacobson radical of E.

For (3.22.3), since I is projectively full, the only ideals that are projectively equivalent to I are the ideals $(I^k)_a$, where k is an arbitrary positive integer, and $(I^k)_a = I^k$, since I is a principal ideal and D is an integrally closed domain. The remainder of (3.22.3) follows from the fact that $I_{(\frac{k}{2})} = (J^k)_a \cap D = (((y^2(y+1))^{\frac{k}{2}})D[y\sqrt{y+1}])_a \cap D$ (by Proposition 2.5.2) and the fact that $D[x] = D + xD$ is a free D-module. □

Remark 3.23.
(3.23.1) Examples similar to Example 3.22 also exist as integral extension domains of a semi-local PID of characteristic zero. For example, with $D = \mathbb{Z}_S$, where $S = \mathbb{Z} \setminus (2\mathbb{Z} \cup 3\mathbb{Z})$, $b = 12$, and $I = b\mathbb{Z}$, consider $J = xD[x]$, where $x = \sqrt{b}$, and let E be the integral closure of $D[x]$ in its quotient field. Then D is a semi-local PID, $d(I) = 1$, and I has the two Rees integers, $e_1 = 1$ associated to the Rees valuation ring D_{3D}, and $e_2 = 2$ associated to the Rees valuation ring D_{2D}. Since $X^2 - 3$ factors as a square over the field $D/2D$, the valuation ring D_{2D} has a unique extension V to the quadratic field extension $\mathbb{Q}(x)$, and the Rees integer of xE with respect to V is 2, so the Rees integers of xE are not all equal to one and xE is not the Jacobson radical of E.
(3.23.2) Analogous examples also exist where one takes pth roots for a prime integer $p > 2$. For example, with D as above, let $b = 54 = 3^3 2$, $I = bD$, and consider $J = xD[x]$, where $D[x] = D[X]/(X^3 - b)$. Then I has two Rees integers, $e_1 = 1$ associated to the Rees valuation ring D_{2D}, and $e_2 = 3$ associated to the Rees valuation ring D_{3D}. Since the polynomial $X^3 - 2$ factors as a cube over the field $D/3D$, the valuation ring D_{3D} has a unique extension V to the field $\mathbb{Q}(x)$. If E is the integral closure of $D[x]$, then the Rees integer of xE with respect to V is 3, and xE is not the Jacobson radical of E.
(3.23.3) For each prime integer $p > 2$, one can make a similar construction. Let $D = \mathbb{Z}_S$, where $S = \mathbb{Z} \setminus (2\mathbb{Z} \cup p\mathbb{Z})$, and let $b = 2p^p$. Let $I = b\mathbb{Z}$, and consider $D[x] = D[X]/(X^p - b)$. Then I has two Rees integers, $e_1 = 1$ associated to the valuation ring D_{2D}, and $e_2 = p$ associated to the valuation ring D_{pD}. Since the polynomial $X^p - 2$ factors as a pth power over the field D/pD,

the valuation ring D_{pD} has a unique extension V to the field $\mathbb{Q}(x)$. If E denotes the integral closure of $D[x]$, then the Rees integer of xE with respect to V is p, and xE is not the Jacobson radical of E.

Concerning the examples in Remark 3.23, it is shown in [5] that there does exist a Dedekind domain E that is a finite integral extension domain of D such that $IE = J^m$ for some positive integer m and for some projectively full radical ideal J of E whose Rees integers are all equal to one. It is also shown in [5] that this holds for I as in Example 3.22 for all fields F of characteristic two.

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